MATH 2040 Linear Algebra II
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Quick Review of Linear Algebra I [1]

In this short note, we list the topics that you are expected to be familiar with before taking MATH 2040. If you find something unfamiliar, please review the corresponding materials of MATH 1030 (or any equivalent courses you have taken).

- Introduction
  - an overview of linear algebra
    - a brief introduction to key concepts/questions which will be covered
  - the geometry of linear equations
    - study 2×2 and 3×3 systems of linear equations
    - understand a linear system as intersection of lines/planes,
    - understand geometrically about different cases of solution types
  - Examples of Gaussian elimination
    - solve a 3×3 system by elimination and back-substitution
    - cover scenarios with one solution, no solution and infinitely many solutions
- Gaussian elimination and matrices
  - Matrix notation and matrix multiplication
    - representing a linear system by vectors and matrices as $Ax = b$
    - matrix-vector multiplication
    - matrix multiplication, its properties and interpretations
    - give examples to show $AB \neq BA$ in general
  - Gaussian elimination in matrix form
    - elementary row operations and their corresponding elementary matrices
    - examples of solving $Ax = b$ by Gaussian eliminations
  - Inverse and Transpose
    - definition of A^{-1} and its relation to solving $Ax = b$
    - properties of inverse, existence and uniqueness, $(AB)^{-1} = B^{-1}A^{-1}$
    - calculate A^{-1} by "Gauss-Jordan method", i.e. $[A|I] \to [I|A^{-1}]$
    - definition of A^t and two basic properties: $(AB)^t = B^tA^t$ and $(A^{-1})^t = (A^t)^{-1}$
- Solving system of linear equations
  - Solving $Ax = 0$ and $Ax = b$
    - column space $C(A)$, nullspace $N(A)$ and their geometric meaning

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    * compute $C(A)$ and $N(A)$ by solving linear systems
    * relationship between the solutions of $Ax = 0$ and $Ax = b$
    * reduced row echelon form (rref), pivot and free variables, rank
    * using rref to express solutions to $Ax = b$ as linear combination of vectors
    * intuitive notion of "dimension" of solution set by counting free variables

- Concepts of linear algebra in $\mathbb{R}^n$
  - Linear independence and span, basis and dimension
    * definition, examples and geometric meaning of linear (in)dependence
    * columns of A are linearly independent iff $N(A) = \{0\}$
    * determine if a given set of vectors in $\mathbb{R}^m$ are linearly independent
    * definition and examples of the span of n vectors in $\mathbb{R}^m$
    * definition of a basis and coordinate system
    * dimension as the number of vectors in a basis
    * dimensions of $C(A)$ and $N(A)$, the dimension formula
  - Linear transformation
    * matrix multiplication as linear transformation
    * the matrix of scaling, rotation, reflection and projection in $\mathbb{R}^2$
    * finding the matrix of a geometric linear transformation
    * change of basis formula, similar matrices

- Inner product and orthogonality in $\mathbb{R}^n$
  - Orthogonal vectors and subspaces
    * inner product and length of vectors
    * definition of orthogonal/orthonormal vectors, Pythagoras theorem
    * prove that a set of non-zero orthogonal vectors are linear independent
    * orthogonality of two subspaces, definition and properties of orthogonal complement
    * (optional) $N(A) = (C(A^t))^\perp$ and $C(A) = (N(A^t))^\perp$
  - Orthogonal projections
    * angle between two vectors, cosine law
    * projection onto a line, Cauchy-Schwarz inequality
    * orthonormal basis and Gram-Schmidt process

- Determinants
  - Properties of determinants
    * use 2×2 determinant to explain the three defining properties of $\det A$: $\det I = 1$, determinant changes sign when two rows are exchanged, and linearity for each row
    * $\det A = 0$ if two rows of A are equal
    * $\det A$ remains unchanged when a multiple of a row is added to another

    * $\det A = 0$ if A has a row of zeros
    * $\det A$ is the product of its diagonals when A is triangular
    * A is invertible iff $\det A \neq 0$
    * $\det AB = \det A \cdot \det B$, $\det A^{-1} = 1/\det A$
    * $\det A^t = \det A$
  - Formula for computing determinants
    * explicit formula for 3×3 determinant
    * compute $\det A$ by elimination until triangular
    * compute $\det A$ by cofactor expansion
  - Applications of determinants
    * compute A^{-1} using cofactors and determinant
    * Cramer's rule
    * 2×2 determinant as (signed) area of parallelogram

- Diagonalization of matrices
  - Eigenvalue and eigenvectors
    * definitions of eigenvalues and eigenvectors
    * compute eigenvalues from characteristic polynomial
    * compute eigenvectors by solving $(A - \lambda I)x = 0$
  - Diagonalization of a matrix
    * existence of eigenbasis and diagonalizability
    * Any $n \times n$ matrix with n distinct eigenvalues is diagonalizable
    * an example of non-diagonalizable matrix
    * algebraic and geometric multiplicity, a brief discussion on diagonalizability
    * a square matrix is invertible $\Leftrightarrow$ 0 is not an eigenvalue
    * finding an invertible matrix Q such that $Q^{-1}AQ$ is diagonal